An Envy-Free Cake Division Protocol

Steven J. Brams and Alan D. Taylor

Our starting point is the well-known parental solution to the problem of dividing a cake between two children so that each child thinks he or she has been treated fairly: The parent instructs one of the children to cut the cake into two pieces in any way he desires. The other child is then instructed to choose whichever piece she prefers. This two-step sequence of instructions, known as "cut-and-choose," provides a simple example of the kind of game-theoretic algorithm that Even and Paz [**11**] call a "protocol."

Associated with the cut-and-choose protocol is a natural strategy for each child: The first child cuts the cake into two pieces that he considers to be equal, and the second child chooses a piece that she considers to be at least as large as the other piece. Notice that each child's strategy guarantees him or her "satisfaction," regardless of what the other child does.

The general version of this problem involves n people ("players"), each of whom has his or her preferences over subsets of the cake given by a probability measure.[1] An allocation of the cake among the players is said to be *proportional* if each player receives a piece of size at least $1/n$ (in his or her own measure), and it is said to be *envy-free* if each player receives a piece he or she would not swap for that received by any other player. It is easy to see that an envy-free allocation is proportional, but the converse fails unless $n = 2$. Thus, for example, every one of three players may think his or her piece is at least $1/3$, but a player may think that one of the other players has a larger piece.

The results on proportional and envy-free allocations obtained over the past 50 years tend to fall into one of four classes: (i) Existence Theorems; (ii) Moving-Knife Solutions; (iii) Algorithms; and (iv) Protocols. We say something about each in turn.

Existence theorems, dating back to the 1940s, are often based on some version of Liapounoff's Convexity Theorem [**20**]. Typically, they establish the existence of an ordered partition of the cake corresponding to an envy-free allocation, often with some additional property such as: all the measures of all the pieces are exactly $1/n$ [**21**]; or the pieces are connected sets [**27** and **31**]; or the allocation is also Pareto-optimal [**30** and **4**]. In the words of Rebman [**24**, p. 33], however, these results provide "no clue as to how to accomplish such a wonderful partition."

There are two well-known moving-knife solutions. The first is due to Dubins and Spanier [**10**] and is a moving-knife version of the Banach-Knaster last-

[1] If one wants to abandon the cake metaphor, and literally work with arbitrary probability measures on some set C, then even cut-and-choose can fail. For example, if both children have their preferences given by the same 0-1 valued measure, i.e., the same ultrafilter, then both will want the same piece regardless of how the cake is divided.

For everything we do in the present paper, it suffices to assume that our measures are all defined on the same algebra $\mathscr{A}$ of subsets of C and satisfy the following two properties: (i) for every set $P \in \mathscr{A}$ and every finite k, P can be partitioned into k sets of equal measure, and (ii) for every $P, Q \in \mathscr{A}$, either P can be trimmed to yield a subset the same size as Q, or vice-versa.

diminisher scheme for n people that we shall present later. (The Dubins-Spanier scheme is easy to describe: a knife is slowly moved along the top of the cake so that all the slices made are parallel. Each player calls "cut" when he or she is willing to take the resulting piece as his or her allocation.) The second is a moving-knife scheme due to Stromquist [27] that yields an envy-free allocation among three people. (This one is not so easy to describe, because envy-freeness is considerably harder to obtain than proportionality.)

Moving-knife schemes, however, are not the step-by-step processes one usually associates with the term "algorithm." A good example of what one would call an algorithm in this context is Woodall's scheme [32] for producing an allocation whereby each participant gets strictly more than $1/n$ of the cake (according to his own measure). This algorithm requires, as part of the input, a piece P of cake and two distinct numbers α and β such that Player 1 thinks the measure of P is α, and Player 2 thinks the measure of P is β. An envy-free version of this algorithm is in **[5]**.

Finally, there is what Even and Paz **[11]** call a "protocol," and this is the only kind of result we are going to analyze in the present paper. Since we will be presenting examples of protocols, as opposed to proving their non-existence, we can afford the same level of informality in the description of what is meant by a protocol as Even and Paz used.

A *protocol* is a computer-programmable interactive procedure that can issue queries to the participants whose answers may affect future decisions. It may issue instructions to the participants such as: "Choose k pieces from among these m pieces" or "Partition this piece into k subpieces." The protocol has no information on the measure of the various pieces as seen by the different participants—this is private information. Moreover, if the participants obey the protocol, then each participant will end up with a piece after finitely many steps.

Still following **[11]**, we define a *strategy* for a participant to be an adaptive sequence of moves consistent with the protocol, which the participants choose sequentially when called upon by the protocol. A protocol is *proportional* if each of the n participants has a strategy that will guarantee him at least $1/n$ of the cake (by his own measure), independently of the other participants' strategies. (Purely for convenience, we will henceforth use only the masculine pronoun.) Departing from **[11]**, we will call a protocol *envy-free* if each of the n participants has a strategy that will guarantee him a piece that is, according to his own measure, at least tied for largest.

A constructive proof of the existence of, say, a proportional protocol involves producing three separate things: the rules of the protocol, a strategy for each of the players, and an argument that the strategies do, in fact, guarantee each player his proportional share. We distinguish rules and strategies by demanding that rules be enforceable by a referee implementing the protocol.

This means that a statement like "Player 1 cuts the cake into n pieces" is an acceptable rule, whereas a statement like "Player 1 cuts the cake into n pieces that he considers to be equal" is not. This is because the latter statement cannot be enforced by the referee, who has no knowledge of Player 1's measure and so cannot tell if the rule has been followed or not.

In presenting protocols, we will separate rules from strategies by placing all strategic aspects in parentheses. This provides one with the option of reading the rules alone in a reasonably smooth way. All arguments that the strategies perform as advertised are placed between steps and labeled as "Aside." For example, in our method of presentation, cut-and-choose becomes:

Cut-and-Choose

Step 1. Player 1 cuts the cake into 2 pieces (that he considers to be the same size).

Step 2. Player 2 chooses a piece (that she considers to be at least tied for largest).

Aside. Clearly, Player 1's strategy guarantees him a piece of size exactly $1/2$ in his measure, while Player 2's strategy guarantees her a piece of size at least $1/2$ in her measure.

The modern era of cake cutting began with Steinhaus' observation "during the war [World War II]" [**25**, p. 102] that the cut-and-choose protocol could be extended to yield a proportional protocol for three players (see [**18**]). He then asked if it could be extended to yield a proportional protocol for the case $n > 3$. (Steinhaus, however, never used the word "protocol.") His question was answered in the affirmative by Banach and Knaster and reported in [**25**] and [**26**]. The Steinhaus and Banach–Knaster protocols introduced two key ideas that would resurface in the envy-free solutions 15 and 50 years later.

The first idea was that of having an initial sequence of steps resulting in only part of the cake's being allocated (to one player in this case). The sequence is then repeated a finite number of times, after which the entire cake has been allocated. The second idea—and perhaps the more important of the two—was that of having a player trim a piece to a smaller size.

Explicit mention of the lack of a *constructive* procedure for producing an envy-free allocation among more than two people dates back at least to Gamow and Stern [**14**]. The first breakthroughs on this problem occurred in the late 1950s and early 1960s, when the protocol solution to the envy-free problem for $n = 3$ was found by John L. Selfridge, and rediscovered independently by John H. Conway. These solutions also involved trimming and an initial allocation of only part of the cake; they were widely disseminated by R. K. Guy and others, and eventually reported by Gardner [**15**], Woodall [**32**], Stromquist [**27**], and Austin [**1**]. The moving-knife solution of Stromquist [**27**] was found two decades later, as was a scheme due to Levmore and Cook [**19**], which can be recast as quite a different moving-knife solution to the envy-free problem when $n = 3$. Still other envy-free moving-knife schemes for three people [**7**] and, more recently, four people [**8**] have been discovered and are summarized in [**6**].

The extension of the Selfridge-Conway protocol to the case of even four people has remained an open, and much-commented upon, problem. See, for example, Gardner [**15**], Rebman [**24**], Stromquist [**27**], Woodall [**31**], [**32**], Bennett *et al* [**3**], Webb [**29**], Hill [**16**], [**17**], and Olivastro [**22**]. In what follows, we solve this problem by producing an envy-free protocol for arbitrary n.

We have chosen a uniform presentation of four protocols that highlights the evolution of two important ideas—namely, trimming, and the use of sequences of partial allocations. Historically, these four protocols arose over a period of 50 years and nicely illustrate how ideas in mathematics are built, one upon another. The protocols we present are:

1. The proportional protocol for $n = 3$ (Steinhaus).
2. The proportional protocol for arbitrary n (Banach-Knaster).
3. The envy-free protocol for $n = 3$ (Selfridge, Conway).
4. The envy-free protocol for arbitrary n.

Before turning to the protocols themselves, we must acknowledge the help of several people. Our interest in fair division was sparked by Olivastro [22]. Valuable mathematical contributions were made by William Zwicker and Fred Galvin. Indeed, the present version of our envy-free protocol owes much to the reworking of an earlier version by Galvin.

Specific observations and comments by David Gale, Sergiu Hart, Theodore Hill, Walter Stromquist, William Webb, and Douglas Woodall also proved helpful. In addition, we have benefited from conversations and correspondence with the following people: Ethan Akin, Julius Barbanel, John Conway, Morton Davis, Karl Dunz, Shimon Even, A. M. Fink, Peter Fishburn, Martin Gardner, Richard Guy, D. Marc Kilgour, Peter Landweber, Jerzy Legut, Herve Moulin, Dominic Olivastro, Barry O'Neill, Philip Reynolds, William Thomson, Hal Varian, Charles Wilson, and H. Peyton Young.

The first protocol we present is a generalization of cut-and-choose to a proportional protocol for three people. This is the one found by Steinhaus during World War II.

The Proportional Protocol for $n = 3$
(Steinhaus, circa 1943)

Step 1. Player 1 cuts the cake into 3 pieces (that he considers to be the same size).

Step 2. Player 2 is given the choice of either passing, i.e., doing nothing (which he does if he thinks 2 or more of the pieces are of size at least $1/3$), or not passing and labeling 2 of the pieces (that he thinks are of size strictly less than $1/3$) as "bad."

Step 3. If Player 2 passed in step 2, then Players 3, 2, and 1, in that order, choose a piece (that they consider to be of size at least $1/3$).

Aside. In this case, each player receives a piece of size at least $1/3$ in his own measure. This is true of: Player 3, because he chooses first; Player 2, because he thinks either 2 or 3 pieces are that large, and so at least one of them will still be available after Player 3 chooses his piece; and Player 1, because he made all 3 pieces of size $1/3$.

Step 4. If Player 2 did not pass at Step 2, then Player 3 is given the same two options that Player 2 had at Step 2. He ignores Player 2's labels.

Step 5. If Player 3 passed in Step 4, then Players 2, 3, and 1, in that order, choose a piece (that they consider to be of size at least $1/3$).

Aside. In this case, as before, each player receives a piece of size at least $1/3$ in his own measure.

Step 6. If Player 3 did not pass at Step 4, then Player 1 is required to take a piece that both Player 2 and Player 3 labelled as "bad."

Aside. Note first that there certainly must be such a piece. At this point, Player 1 has received a piece that he thinks is of size exactly $1/3$, which both Player 1 and Player 2 think is "bad," i.e., of size strictly less than $1/3$.

Step 7. The other two pieces are reassembled, and Player 2 cuts the resulting piece into two pieces (that he considers to be the same size).

Step 8. Player 3 chooses one of the two pieces (that he considers to be at least tied for largest).

Step 9. Player 2 is given the remaining piece.

Aside. This is just cut-and-choose between Players 2 and 3, which ends the protocol.

The second protocol we present followed quickly on the heels of the first. It is the Banach-Knaster protocol, offered in response to Steinhaus' question of whether his result could be extended from 3 to n people. Note here the introduction of the idea of trimming, which will be further exploited in both of the upcoming envy-free protocols.

Proportional Protocol for Arbitrary n
(Banach-Knaster, circa 1944)

Step 1. Player 1 cuts a piece P_1 (of size $1/n$) from the cake.
Step 2. Player 2 is given the choice of either passing (which he does if he thinks P_1 is of size less than $1/n$), or trimming a piece from P_1 to create a smaller piece (that he thinks is of size exactly $1/n$). The piece P_1, now perhaps trimmed, is renamed P_2. The trimmings are set aside.
Step 3. For $3 \leq i \leq n$, Player i takes the piece P_{i-1} and proceeds exactly as Player 2 did in Step 2, with the resulting piece now called P_i.
Aside. For $1 \leq i \leq n$, Player i thinks that P_i is of size less than or equal to $1/n$. We also have that $P_1 \supset \cdots \supset P_n$. Thus, every player thinks P_n is of size at most $1/n$.
Step 4. The last player to trim the piece, or Player 1 if no one trimmed it, is given P_n.
Aside. The player receiving P_n thinks it is of size exactly $1/n$.
Step 5. The trimmings are reassembled, and Steps 1–4 are repeated for the remainder of the cake, and with the remaining $n - 1$ players in place of the original n players.
Aside. The player who gets a piece at this second stage is getting exactly $1/(n - 1)$ of the remainder of the cake; he, and everyone else, thinks this remainder is of size at least $(n - 1)/n$. Hence, he thinks his piece is of size at least $1/n$.
Step 6. Step 5 is iterated until there are only 2 players left. The last 2 players use cut-and-choose.
Aside As before, each player receives a piece that he thinks is of size at least $1/n$.
This ends the protocol.

The next protocol we present is the Selfridge-Conway envy-free protocol for the case $n = 3$. (There are slight differences in the presentations of Selfridge and Conway; we follow the latter.) This protocol involves an elegant combination of the trimming idea introduced by Banach-Knaster and the basic framework that Steinhaus used. It also introduces the important notion of one player's having an "irrevocable advantage" over another player, following a partial allocation.

Envy-Free Protocol for $n = 3$
(Selfridge, Conway, circa 1960)

Step 1. Player 1 cuts the cake into 3 pieces (that he considers to be the same size).
Step 2. Player 2 is given the choice of either passing (which he does if he thinks two or more pieces are tied for largest), or trimming a piece from (the largest) one of the three pieces (to create a tie for largest). If Player 2 trimmed a piece, then the trimmings are named L, for "leftover," and set aside.

Step 3. Players 3, 2, and 1, in that order, choose a piece (that they consider to be at least tied for largest) from among the 3 pieces, one of which may have been trimmed in Step 2. If Player 2 did not pass in Step 2, then he is required to choose the piece he trimmed if Player 3 did not.

Aside. Notice that only part of the cake has been allocated. This yields a partition $\{X_1, X_2, X_3, L\}$ of the cake such that $\{X_1, X_2, X_3\}$ is an envy-free partial allocation. The lack of envy is true of: Player 3, because he chooses first; Player 2, because he made at least two pieces tied for largest, and so at least one of them will still be available after Player 3 chooses his piece; and Player 1, because he made all three pieces of size $1/3$, and the trimmed one has definitely been taken by either Player 3 or Player 2.

Step 4. If Player 2 passed at Step 2, we are done. Otherwise, either Player 2 or Player 3 received the trimmed piece, and the other received an untrimmed piece. Whichever player received the *un*trimmed piece now divides L into 3 pieces (that he considers to be the same size). Call this player the "cutter" and the other the "non-cutter."

Aside. We will refer to Player 1 as having an *irrevocable advantage* over the non-cutter. The point is that, since the non-cutter received the trimmed piece, Player 1 will not envy the non-cutter, *regardless* of how L is later divided among the three.

Step 5. The three pieces into which L is divided are now chosen by the players in the order: non-cutter first; Player 1 second; cutter third. (Each chooses a piece at least tied for largest among those available to him when it is his turn to choose.)

Aside. At this point, the entire cake has been allocated. Since the non-cutter chooses his piece of L first, he experiences no envy. Player 1 does not envy the non-cutter, since he had an irrevocable advantage over him, and Player 1 does not envy the cutter, because he is choosing his piece of L before the cutter does. Finally, the cutter experiences no envy since he divided L into three equal pieces.

This ends the protocol.

The final protocol we present is our envy-free protocol for an arbitrary number of players. This result was announced in [**9**, **12**, **13**, **23**]. A brief discussion of some important differences between this protocol and the three earlier ones, and a couple of important open questions, follow.

The central feature of our envy-free protocol, like that for the $n = 3$ protocol, is that players trim pieces of the cake to create ties, rendering them indifferent among these pieces. When $n > 3$, however, one needs to start the trimming and choosing process—leading to an envy-free partial allocation—with *more* pieces than there are players.

As an informal illustration of how to achieve an envy-free *partial* allocation, suppose there are four people. Have Player 1 cut the cake into 5 equal pieces. Player 2 then trims 2 pieces, creating a 3-way tie for largest. Player 3 then trims 1 piece, creating a 2-way tie for largest. The players now choose in the order: Player 4, Player 3, Player 2, Player 1, with the middle two players required to take a piece they trimmed if one is available. Clearly, each player thinks his piece is at least tied for largest. The burden of our demonstration of the n-person envy-free protocol is to show that a full allocation of the entire cake can be accomplished in a *finite* number of steps.

For simplicity, we will present only the $n = 4$ version of the envy-free protocol. The extension to arbitrary n is fairly straightforward and left to the reader. In outline form, the protocol goes as follows:

One player (chosen here to be Player 2 for later notational simplicity), cuts the cake into 4 equal pieces, hands these out, and asks if anyone objects. If, say, Player 1 objects, then Players 1 and 2 (alone) go through several steps which yield six sets (the Ys and Zs in Step 7 below) to be used as a starting partition (in place of the five equal pieces) for the kind of trim-and-choose sequence among all four players that we illustrated two paragraphs earlier. This trim-and-choose sequence is repeated again and again until we arrive at a partial allocation in which Player 1 has an irrevocable advantage over Player 2 (the "aside" after Step 15 below). From this point on, we never have to worry about Player 1's objecting because of envy for Player 2. Repeating this at most once for each pair of players results in an envy-free allocation of the entire cake after finitely many steps.

Envy-Free Protocol for Arbitrary n
(the $n = 4$ version)
(1992)

Step 1. Player 2 cuts the cake into 4 pieces (that he considers to be the same size), keeps one piece, and hands one piece to each player.

Step 2. Each of the other three players is asked whether or not he objects to this allocation. (A player objects iff he envies some other player.)

Step 3. If no one objects, then each keeps the piece he was given in Step 1, and we are done.

Step 4. Otherwise, we choose the smallest i so that Player i objected. For notational simplicity, assume $i = 1$. Player 1 now chooses a piece originally given to some other player (whom he envied) and calls that piece A. The piece originally given to Player 1 is called B.

Aside. Once we have A and B, the other two pieces in the allocation from Step 1 are reassembled. That part of the cake will be allocated later. Note that Player 1 thinks A is larger than B. Player 2 thinks A and B are the same size.

Step 5. Player 1 now names a positive integer $r \geq 10$ (chosen so that, for any partition of A into r sets, Player 1 will prefer A, even with the 7 smallest—according to Player 1—pieces in the partition of A removed, to B).

Aside. Player 1 can easily choose such an r. That is, the union of the 7 smallest pieces is certainly no larger than 7 times the average size of all r pieces. Hence, Player 1 simply chooses r large enough so that $7\mu(A)/r < \mu(A) - \mu(B)$, where μ is his measure.

Step 6. Player 2 now partitions A into exactly r sets (that he considers to be the same size), and does the same to B.

Step 7. Player 1 chooses (the smallest) 3 sets from the partition of B and names these Z_1, Z_2, Z_3. He also chooses either (the largest) 3 sets from the partition of A (if he thinks these are all strictly larger than all the Zs), and trims at most 2 of these (to the size of the smallest among the three), or he partitions (the largest) one of the sets in the partition of A into 3 pieces (that he considers to be the same size). In either case, he names these Y_1, Y_2, Y_3.

Aside. Player 1's strategy in Step 7 guarantees that he will think all three Ys are the same size, and each strictly larger than all three Zs. This is true even if he chooses the second option.[2] Player 2 thinks all three Zs are the same size, and each is at least as large as all three Ys.

Step 8. Player 3 takes the collection of 6 pieces, and either passes (if he thinks there already is at least a 2-way tie for largest), or trims (the largest) one of these (to the size of the next largest), thus creating at least a 2-way tie for largest).

Step 9. Players 4, 3, 2, and 1, in that order, choose a piece from among the 6 Ys and Zs as modified in Step 8 (that they consider to be largest or tied for largest), with Player 3 required to take the piece he trimmed if it is available. Player 2 must choose Z_1, Z_2, or Z_3. Player 1 must choose Y_1, Y_2, or Y_3.

Aside. This yields a partition $\{X_1, X_2, X_3, X_4, L_1\}$ of the cake such that $\{X_1, X_2, X_3, X_4\}$ is an envy-free partial allocation, and L_1 is the leftover piece. Moreover, Player 1 thinks his piece X_1 is *strictly larger* —say by ε—than Player 2's piece X_2.

Step 10. Player 1 names a positive integer s (chosen so that $[4\mu_1(L_1)/5]^s < \varepsilon$, where μ_1 is Player 1's measure).

Aside. The integer s specifies how many times the players will iterate the basic trim-and-choose sequence to follow. Notice that if the rules were instead to allow the iterations to continue until Player 1 said "stop" (which he could strategically do at the point at which he thinks the leftover crumb is smaller than the advantage he has over Player 2), then there is no guarantee that a strategically misguided Player 1 would not keep the game going forever.

Step 11. Player 1 cuts L_1 into 5 pieces (that he considers to be the same size).

Step 12. Player 2 takes the collection of 5 pieces, selects (the largest) 3 pieces, and trims (the largest) 2 or fewer of these (to the size of the smallest, thereby creating at least a 3-way tie for largest).

Step 13. Player 3 takes the collection of 5 pieces, perhaps trimmed in step 12, selects (the largest) two, and trims, if he wants to, (the largest) one of these (to the size of the smallest, thus creating at least a 2-way tie for largest).

Step 14. Players 4, 3, 2, and 1, in that order, choose a piece (that they consider

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[2] The proof runs as follows: We are assuming that both A and B have been partitioned into r pieces, and that B is not only smaller than A but smaller even than A with the smallest 7 pieces of A's partition removed. Arrange the sets in both partitions from largest to smallest as $A_1, A_2, \ldots, A_r$ and $B_1, B_2, \ldots, B_r$. Let μ denote Player 1's measure, and suppose, for contradiction, that both of the following hold:

1. $\mu(B_{r-2}) \geq \mu(A_3)$, which holds if A_1, A_2, and A_3 are *not* all strictly larger than B_{r-2}, B_{r-1}, and B_r.

2. $\mu(B_{r-2}) \geq \mu(A_1)/3$, which holds if A_1 *cannot* be partitioned into 3 sets all larger than B_{r-2}, B_{r-1}, and B_r.

It follows from 1 that:

3. $\mu(B_7 \cup \cdots \cup B_{r-3}) \geq \mu(A_3 \cup \cdots \cup A_{r-7})$, since there are $r-9$ sets in each union, and the smallest one of the Bs is at least as large as the largest one of the As.

It follows from 2 that:

4. $\mu([B_1 \cup B_2 \cup B_3] \cup [B_4 \cup B_5 \cup B_6]) \geq \mu(A_1 \cup A_2)$,
since each of the blocks of 3 Bs is larger than each of the As.

But 3 and 4 clearly demonstrate that:

5. $\mu(B) \geq \mu(A_1 \cup \cdots \cup A_{r-7})$.

This is the desired contradiction since the set on the right is A with the smallest 7 pieces of its partition removed.

to be largest or tied for largest), with Players 3 and 2 required to take a piece they trimmed if one is available.

Step 15. Steps 11–14 are repeated $s - 1$ more times, with each application of these four steps applied to the leftover piece from the preceding application.

Aside. This yields a partition $\{X_1', X_2', X_3', X_4', L_2\}$ of the cake such that $\{X_1', X_2', X_3', X_4'\}$ is an envy-free partial allocation, and such that Player 1 thinks that X_1' is larger than $X_2' \cup L_2$. We now declare that Player 1 has an *irrevocable advantage* over Player 2, and we begin creation of a subset of $\{1,2,3,4\} \times \{1,2,3,4\}$, which we call "$\mathscr{I}\mathscr{A}$" for "irrevocable advantage," by putting $(1,2) \in \mathscr{I}\mathscr{A}$.

Step 16. Player 2 cuts L_2 into 12 pieces (that he considers to be the same size).

Step 17. Each of the other players declares himself to be of type A (if he agrees all the pieces are the same size), or type D (if he disagrees). Player 2 is declared to be of type A.

Step 18. If $D \times A \subset \mathscr{I}\mathscr{A}$, then we give the 12 pieces to the players in A, with each of them receiving the same number of pieces. In this case, we are done.

Step 19. Otherwise, we choose the lexicographically least pair (i, j) from $D \times A$ that is *not* in $\mathscr{I}\mathscr{A}$, and we return to Step 4 with Player i in the role of Player 1, Player j in the role of Player 2, and L_2 in place of the cake.

Step 20. Steps 5–18 are repeated.

Aside. Each time we pass through Step 15, we add an ordered pair to $\mathscr{I}\mathscr{A}$. Notice that since $D \times A \subset \{1,2,3,4\} \times \{1,2,3,4\}$, and $\mathscr{I}\mathscr{A} \subset \{1,2,3,4\} \times \{1,2,3,4\}$, we must have $D \times A \subset \mathscr{I}\mathscr{A}$ after at most 16 iterations. At this point, we conclude at Step 18 with an envy-free division of the entire cake.

This ends the protocol.

There is an important way (pointed out to us by several people) that the envy-free protocol for even $n = 4$ differs from the envy-free protocol for $n = 3$: For $n = 3$, the number of cuts needed is at most 5, *regardless* of what the measures are. For $n = 4$, the number of cuts needed can be made arbitrarily large by a suitable choice of the four measures (although the moving-knife solution **[8]** for the four-person problem gives a bounded number of cuts). This raises:

Question 1. Is there a *bounded* envy-free protocol for $n = 4$ or $n > 4$?

There is another slightly more subtle (and perhaps related) way in which the envy-free protocol differs from the others: The three earlier ones also work in the context of what are called "CD preference relations" in **[2]**. (A CD preference relation is a complete, reflexive, transitive binary relation that satisfies a partitioning postulate, a trimming postulate, and a weak additivity postulate.) The envy-free protocol, on the other hand, requires what is call an "Archimedian CD preference relation" in **[2]**. The main result in **[2]** is the fact that a CD preference relation is induced by a finitely additive measure in the obvious way iff it is Archimedian. This raises:

Question 2. Is there an envy-free protocol for $n = 4$ or $n > 4$ that works in the context of *non-Archimedian* preference relations?

It turns out that techniques similar to those used in the n-person envy-free protocol can also be used to solve the "chores" problem **[15]**, wherein each player wants to minimize the amount of cake he or she receives. This and related questions (e.g., the Pareto-optimality of allocations) are discussed in [**6**].

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Department of Politics
New York University
New York, NY 10003

Department of Mathematics
Union College
Schenectady, NY 12308